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LETTER TO THE EDITOR

Invariance and integrability: Hénon–Heiles and two coupled quartic anharmonic oscillator systems

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Abstract. It is shown that the Hénon–Heiles and two coupled quartic anharmonic oscillator systems possess non-trivial generalised Lie symmetries for specific sets of parametric values, for which second integrals of motion can also be constructed directly using Noether's theorem, thereby establishing their complete integrability.

In recent years, considerable effort has been put into identifying integrable dynamical systems having both finite and infinite degrees of freedom by employing different techniques. It is known that the invariance of evolution equations under Lie transformations and generalisations lead to the integrability of the systems (Anderson and Ibragimov 1979, Bluman and Cole 1974, Ovjannikov 1982, Lutzky 1978, 1979, Lakshmanan and Kaliappan 1983). Recently, Lutzky (1979) has shown that the integrals of motion of the finite-dimensional Lagrangian systems can be related to the infinitesimal symmetries under extended Lie transformations involving velocity terms (see also Sarlet and Cantrijn 1981, Prince 1983). In particular, he considered the simple harmonic oscillator problem and derived the associated symmetries having velocity dependent terms. In this letter, we consider the Hénon–Heiles and two coupled quartic anharmonic oscillator systems and show the existence of non-trivial generalised Lie symmetries for specific sets of parametric values at which the so-called Painlevé property is also satisfied and for which integrals of motion can be found by direct methods (Chang *et al* 1982, Lakshmanan and Sahadevan 1985). We also present the second integrals of motion for each set of Lie symmetries by using Noether's theorem.

We consider a Lagrangian system with two degrees of freedom having the form

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-V(x,y) \qquad (\cdot = \mathrm{d}/\mathrm{d}t). \tag{1}$$

The associated Euler–Lagrange equations of motion are

$$\ddot{x}=\partial L/\partial x=\alpha_1(x,y) \qquad \ddot{y}=\partial L/\partial y=\alpha_2(x,y). \tag{2}$$

For equations (2) to be invariant under infinitesimal transformations

$$x\to X=x+\varepsilon\eta_1(t,x,y,\dot{x},\dot{y}) \tag{3a}$$

$$y\to Y=y+\varepsilon\eta_2(t,x,y,\dot{x},\dot{y}) \tag{3b}$$

$$t\to T=t+\varepsilon\xi(t,x,y,\dot{x},\dot{y}) \qquad \varepsilon\ll 1 \tag{3c}$$

we require the following invariance conditions (Lutzky 1979) to be satisfied:

$$\ddot{\eta}_1-\dot{x}\ddot{\xi}-2\dot{\xi}\alpha_1=E(\alpha_1) \tag{4a}$$

$$\ddot{\eta}_2-\dot{y}\ddot{\xi}-2\dot{\xi}\alpha_2=E(\alpha_2) \tag{4b}$$

where the infinitesimal operator E is given by

$$E=\frac{\partial}{\partial t}+\eta_1\frac{\partial}{\partial x}+\eta_2\frac{\partial}{\partial y}+(\dot{\eta}_1-\dot{\xi}\dot{x})\frac{\partial}{\partial \dot{x}}+(\dot{\eta}_2-\dot{\xi}\dot{y})\frac{\partial}{\partial \dot{y}}. \tag{5}$$

In general the non-linear coupled equations (4) form an incomplete system in η_1, η_2 and ξ. Therefore, in order to solve (4), we have to assume specific forms for η_1, η_2 and ξ. Obviously, one such choice is $\xi=$ constant, $\eta_1=\eta_2=0$, from which we may infer that the Hamiltonian H is a constant of motion (see below). To find the existence of other non-trivial symmetries, we may assume η_1, η_2 and ξ to be polynomials in the velocities $\dot{x}$ and $\dot{y}$ and then determine the t, x, y dependence consistently. As an example, we consider the linear form

$$\xi=a_1+a_2\dot{x}+a_3\dot{y} \qquad \eta_1=b_1+b_2\dot{x}+b_3\dot{y} \qquad \eta_2=c_1+c_2\dot{x}+c_3\dot{y} \tag{6}$$

where a_i, b_i and c_i are functions of (t, x, y) only. Making use of (6) in (4) and equating the various coefficients of $\dot{x}^m\dot{y}^n$, $m, n=1, 2, 3, 4$, to zero, we obtain a system of overdetermined partial differential equations:

$$a_{2xx}=0 \qquad 2a_{2xy}+a_{3xx}=0 \qquad a_{2yy}+2a_{3xy}=0 \qquad a_{3yy}=0 \tag{7}$$

$$\begin{gathered} b_{2xx}-(a_{1xx}+2a_{2xt})=0 \qquad (2b_{2xy}+b_{3xx})-2(a_{1xy}+a_{2yt}+a_{3xt})=0 \\ (b_{2yy}+2b_{3xy})-(a_{1yy}+2a_{3yt})=0 \qquad b_{3yy}=0 \end{gathered} \tag{8a}$$

$$\begin{gathered} (b_{1xx}+2b_{2xt})-(2a_{1xt}+a_2\alpha_{1x}+5\alpha_1a_{2x}+a_3\alpha_{2x}+a_{2tt}+2a_2a_{3x}+\alpha_2a_{2y})=0 \\ 2(b_{1xy}+b_{2yt}+b_{3xt})-(2a_{1yt}+a_2\alpha_{1y}+4\alpha_1a_{2y}+a_3\alpha_{2y}+a_{3tt}+3\alpha_2a_{3y}+3\alpha_1a_{3x})=0 \\ b_{1yy}+2b_{3yt}-2\alpha_1a_{3y}=0 \end{gathered} \tag{8b}$$

$$\begin{gathered} 2b_{1xt}+b_2\alpha_{1x}+3\alpha_1b_{2x}+b_3\alpha_{2x}+b_{2tt}+2\alpha_2b_{3x}+\alpha_2b_{2y} \\ -(a_{1tt}+4\alpha_1a_{2t}+\alpha_2a_{1y}+2\alpha_2a_{3t}+3\alpha_1a_{1x})-(b_2\alpha_{1x}+c_2\alpha_{1y})=0 \\ 2b_{1yt}+b_2\alpha_{1y}+2\alpha_1b_{2y}+b_3\alpha_{2y}+b_{3tt}+3\alpha_2b_{3y}+\alpha_1b_{3x} \\ -2\alpha_1(a_{1y}+a_{3t})-(b_3\alpha_{1x}+c_3\alpha_{1y})=0 \end{gathered} \tag{8c}$$

$$b_{1tt}+2\alpha_1b_{2t}+2\alpha_2b_{3t}+\alpha_1b_{1x}+\alpha_2b_{1y}-2\alpha_1(a_{1t}+\alpha_1a_2+\alpha_2a_3)-(b_1\alpha_{1x}+c_1\alpha_{1y})=0 \tag{8d}$$

$$\begin{gathered} c_{2xx}=0 \qquad c_{2yy}+2c_{3xy}-2(a_{1xy}+a_{2yt}+a_{3xt})=0 \\ 2c_{2xy}+c_{3xx}-(a_{1xx}+2a_{2xt})=0 \\ c_{3yy}-(a_{1yy}+2a_{3yt})=0 \end{gathered} \tag{9a}$$

$$\begin{gathered} c_{1xx}+2c_{2xt}-2\alpha_2a_{2x}=0 \\ 2(c_{1xy}+c_{2yt}+c_{3xt})-(2a_{1xt}+a_2\alpha_{1x}+3\alpha_1a_{2x}+a_3\alpha_{2x}+a_{2tt}+4\alpha_2a_{3x}+3\alpha_2a_{2y})=0 \\ c_{1yy}+2c_{3yt}-(2a_{1yt}+a_2\alpha_{1y}+2\alpha_1a_{2y}+a_3\alpha_{2y}+a_{3tt}+5\alpha_2a_{3y}+\alpha_1a_{3x})=0 \end{gathered} \tag{9b}$$

$$\begin{gathered} (2c_{1xt}+c_2\alpha_{1x}+3\alpha_1c_{2x}+c_3\alpha_{2x}+c_{2tt}+2\alpha_2c_{3x}+\alpha_2c_{2y}) \\ -2\alpha_2(a_{1x}+a_{2t})-(b_2\alpha_{2x}+c_2\alpha_{2y})=0 \\ (2c_{1yt}+c_2\alpha_{1y}+2\alpha_1c_{2y}+c_3\alpha_{2y}+c_{3tt}+3\alpha_2c_{3y}+\alpha_1c_{3x}) \\ -(a_{1tt}+2\alpha_1a_{2t}+4\alpha_2a_{3t}+\alpha_1a_{1x}+3\alpha_2a_{1y})-(b_3\alpha_{2x}+c_3\alpha_{2y})=0 \end{gathered} \tag{9c}$$

$$(c_{1tt}+2\alpha_1 c_{2t}+2\alpha_2 c_{3t}+\alpha_1 c_{1x}+\alpha_2 c_{1y})$$
$$-2\alpha_2(a_{1t}+\alpha_1 a_2+\alpha_2 a_3)-(b_1\alpha_{2x}+c_1\alpha_{2y})=0 \tag{9d}$$

where subscripts denote partial derivatives. By successively solving equations (7)-(9) with the help of the equation of motion (2) we can find the explicit forms of ξ, η_1 and η_2. Specific results are as follows.

(*a*) *Hénon-Heiles system*

The Lagrangian:

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-\tfrac{1}{2}(Ax^2+By^2)-(Dx^2y-\tfrac{1}{3}Cy^3). \tag{10}$$

The Euler-Lagrange equations are

$$\ddot{x}=-Ax-2Dxy \qquad \ddot{y}=-By-Dx^2+Cy^2 \tag{11}$$

where A, B, C and D are parameters. Now, from equation (7), we write

$$a_2=a_{20}y^2+a_{21}xy+a_{22}x+a_{23}y+a_{24} \tag{12}$$

$$a_3=-a_{21}x^2-a_{20}xy+a_{31}x+a_{32}y+a_{33} \tag{13}$$

where the a_{ij} are functions of t to be determined. Also from the last and first equations of (8*b*) and (9*b*), we obtain for the system (11)

$$b_1=-\tfrac{2}{3}Db_{10}(a_{32}-a_{20}x)xy^3+b_{11}y^2+b_{12}y+b_{13} \tag{14}$$

$$c_1=-\tfrac{1}{6}Dc_{10}(a_{21}y+a_{22})x^4+c_{11}x^3+c_{12}x^2+c_{13}x+c_{14} \tag{15}$$

where the b_{1i} and c_{1i} are functions of (t, x) and (t, y) respectively. Proceeding further, from (8*a*) and (9*a*) we get

$$b_3=b_{30}y+b_{31} \qquad c_2=c_{20}x+c_{21}.$$

Here again the coefficients b_{3i} and c_{2i} are functions of (t, x) and (t, y) respectively. Making use of equations (12)-(15) in equations (8*b*) and (9*b*), we have

$$-4Db_{10}(a_{32}-a_{20}x)xy+2b_{11}+2\dot{b}_{30}+2(A+2Dy)(a_{32}-a_{20})x=0 \tag{16a}$$

$$-2Dc_{10}(a_{21}y+a_{22})x^2+6c_{11}x+2c_{12}+2\dot{c}_{20}+2(By+Dx^2-Cy^2)(a_{21}y+a_{22})=0. \tag{16b}$$

Equating the different powers of x and y to zero, we get

$$\begin{array}{llll} a_{20}=a_{32} & a_{21}=-a_{22} & b_{10}=\text{constant} & b_{11}=-\dot{b}_{30} \\ c_{11}=0 & c_{10}=\text{constant} & c_{12}=-\dot{c}_{20}. & \end{array} \tag{17}$$

Continuing this procedure for the remaining equations in (8) and (9) together with equations (12)-(17), we find that the consistency conditions hold for the following parametric choices (excluding the trivial case $C=D=0$):

$$A=B \qquad C=-D \tag{18a}$$

and

$$\begin{array}{ll} a_1=a_2=a_3=0 & b_1=b_2=c_1=c_3=0 \\ b_3=\text{constant} & c_2=\text{constant}. \end{array} \tag{18b}$$

$$A, B \text{ arbitrary} \qquad C=-6D \tag{19a}$$

and

$$a_1=a_2=a_3=0 \qquad b_1=0 \qquad b_2=2(4A-B)-8Dy \qquad b_3=4Dx$$
$$c_1=0 \qquad c_2=4Dx \qquad c_3=0. \tag{19b}$$

Repeating the above analysis by now considering ξ, η_1 and η_2 as a cubic polynomial in $\dot{x}$ and $\dot{y}$ with

$$\xi=\sum_{i,j=0}^{3} a_{ij}\dot{x}^i\dot{y}^j \qquad \eta_1=\sum_{i,j=0}^{3} b_{ij}\dot{x}^i\dot{y}^j$$
$$\eta_2=\sum_{i,j=0}^{3} c_{ij}\dot{x}^i\dot{y}^j \qquad (i+j\leqslant 3) \tag{20}$$

where the a_{ij}, b_{ij} and c_{ij} are functions of (t, x, y), we find that, for the system (11), for an additional parametric choice:

$$16A=B \qquad C=-16D \tag{21}$$

the consistency conditions are fulfilled. Here

$$a_{ij}=0 \qquad i,j=0,1,2,3 \tag{22a}$$

$$b_{00}=0 \qquad b_{10}=2(A+2Dy)x^2 \qquad b_{01}=-\tfrac{4}{3}Dx^3 \qquad b_{30}=4 \qquad c_{10}=-\tfrac{4}{3}Dx^3 \tag{22b}$$

and the other coefficients vanish.

Recently, Fordy (1983) has also reported that non-trivial Hamiltonian symmetries exist exactly for the same parametric choices isolated above ((18*a*), (19*a*) and (21)) by deriving the associated commuting Hamiltonian flows.

(*b*) *Two coupled quartic anharmonic oscillators*

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-(Ax^2+By^2+\alpha x^4+\beta y^4+\delta x^2y^2) \tag{23a}$$

$$\ddot{x}=-2Ax-4\alpha x^3-2\delta xy^2 \qquad \ddot{y}=-2By-4\beta y^3-2\delta x^2 y \tag{23b}$$

where A, B, α, β and δ are parameters. Proceeding in an exactly similar fashion as that of the Hénon-Heiles system above, we verify that there exist four parametric choices possessing non-trivial symmetries. We summarise these results in table 1.

Having obtained the explicit forms of symmetries ξ, η_1 and η_2, we proceed to find the associated conserved quantities which are in involution. For this purpose, we make use of the fact that, given the infinitesimal symmetries η_1, η_2 and ξ and the Lagrangian L, the conserved quantity, if it exists, may be written as (Noether's theorem)

$$I=(\xi\dot{x}-\eta_1)\,\partial L/\partial\dot{x}+(\xi\dot{y}-\eta_2)\,\partial L/\partial\dot{y}-\xi L+f \tag{24}$$

where f is to be determined from the equation

$$E\{L\}+\dot{\xi}L=\dot{f}. \tag{25}$$

For a detailed discussion on the applicability of Noether's theorem to dynamical symmetries, we may refer to Sarlet and Cantrijn (1981). Then the second integrals of motion take the following forms.

Table 1. Infinitesimal symmetries of Hénon-Heiles and two coupled quartic anharmonic oscillator (AHO) systems.

System	Parametric restrictions	Infinitesimal symmetries		
		ξ	η_1	η_2
Hénon-Heiles	(a) $A=B$, $C=-D$	0	$k\dot{y}$	$k\dot{x}$
	(b) A, B arbitrary, $C=-6D$	0	$4D(x\dot{y}-2\dot{x}y)$ $+2(4A-B)\dot{x}$	$4Dx\dot{x}$
	(c) $16A=B$, $C=-16D$	0	$4\dot{x}^3+4(A+2Dy)$ $x^2\dot{x}-\frac{4}{3}Dx^3\dot{y}$	$-\frac{4}{3}Dx^3\dot{x}$
Two coupled AHO	(a) A, B arbitrary, $\alpha=\beta$, $\delta=2\alpha$	0	$2y(\dot{x}y-x\dot{y})$ $+\frac{2}{\alpha}(B-A)\dot{x}$	$2x(x\dot{y}-y\dot{x})$
	(b) $A=B$, $\alpha=\beta$, $\delta=6\alpha$	0	$k\dot{x}$	$k\dot{y}$
	(c) $A=4B$, $\alpha=16\beta$, $\delta=12\beta$	0	$x\dot{y}$	$y\dot{x}-2x\dot{y}$
	(d) $A=4B$, $\alpha=8\beta$, $\delta=6\beta$	0	$8\beta(y\dot{x}-2x\dot{y})y^3$	$4\dot{y}^3+8(B+\beta y^2+6\beta x^2)y^2\dot{y}$ $-16\beta xy^3\dot{x}$

(a) *Hénon-Heiles system*

$$I_a=\dot{x}\dot{y}+Axy+\tfrac{1}{3}Dx^3+Dxy^2 \qquad (26a)$$

$$I_b=-4(y\dot{x}-x\dot{y})\dot{x}+4Ax^2y+x^4+4x^2y^2+(4A-B)(\dot{x}^2+Ax^2) \qquad (26b)$$

$$I_c=\dot{x}^4+2(A+2Dy)x^2\dot{x}^2-\tfrac{4}{3}Dx^3\dot{x}\dot{y}+A^2x^4-\tfrac{4}{3}D(A+Dy)x^4y-\tfrac{2}{9}D^2x^6. \qquad (26c)$$

(b) *Two coupled quartic anharmonic oscillators*

$$I_a=(x\dot{y}-y\dot{x})^2+\frac{2}{\alpha}(B-A)(\tfrac{1}{2}\dot{x}^2+Ax^2+\alpha x^4+\alpha x^2y^2) \qquad (27a)$$

$$I_b=\dot{x}\dot{y}+2Axy+4\alpha xy(x^2+y^2) \qquad (27b)$$

$$I_c=-x\dot{y}^2+y\dot{x}\dot{y}+2(B+4\beta x^2+2\beta y^2)xy^2 \qquad (27c)$$

$$I_d=\dot{y}^4+4(B+6\beta x^2+\beta y^2)y^2\dot{y}^2-16\beta xy^3\dot{x}\dot{y}+4\beta y^4\dot{x}^2$$
$$+4B(B+4\beta x^2+2\beta y^2)y^4+4\beta^2(2x^2+y^2)^2y^4. \qquad (27d)$$

These are indeed exactly the same cases which are found to be integrable through the Painlevé analysis and by a direct search for the second integral of motion (Chang *et al* 1982, Lakshmanan and Sahadevan 1985). Here, we have succeeded in obtaining them through the invariance analysis.

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